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FAST TRACK COMMUNICATION

Solvable nonlinear evolution PDEs in multidimensional space involving trigonometric functions

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Abstract

A *solvable* nonlinear (system of) evolution PDEs in multidimensional space, involving *trigonometric* (or *hyperbolic*) functions, is identified. An *isochronous* version of this (system of) evolution PDEs in multidimensional space is also reported.

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1. Introduction and main results

Recently, certain nonlinear (systems of) evolution PDEs have been investigated [1], whose *solvability* in *multidimensional* space had been pointed out over a dozen years ago [2]. (*Terminology*: a *nonlinear* evolution PDE is considered *solvable* if its solution can be reduced to solving *linear* PDEs (themselves generally solvable by standard techniques, such as the Fourier transform and quadratures), and possibly also to solving (explicitly known) algebraic or transcendental equations, including the inversion of (explicitly known) transformations—but without having to solve any *nonlinear differential* equation.) The purpose and scope of this paper is to extend these results to a more general class of nonlinear evolution PDEs in *multidimensional* space, involving *trigonometric* (or *hyperbolic*) functions. This extension is performed via an approach analogous to that described in section 2.3.5 of [3]. An extension involving elliptic functions is also feasible, via an approach analogous to that of [4]; but these results are sufficiently different to suggest treating them separately [5].

The (system of) nonlinear evolution PDEs treated in this paper read as follows:

$$\alpha\psi_{n,t} + \beta\psi_{n,tt} + \eta\Delta\psi_n = \lambda + 2c\sum_{m=1,m\neq n}^{N}\cot[c(\psi_n-\psi_m)][\beta\psi_{n,t}\psi_{m,t} + \eta(\vec{\nabla}\psi_n)\cdot(\vec{\nabla}\psi_m)+\mu]. \qquad (1)$$

Notation: N is a positive integer, $N \geqslant 2$; indices such as n and m run from 1 to N unless otherwise indicated; the N dependent variables are $\psi_n \equiv \psi_n(\vec{r}, t)$; the independent variable t denotes the *time*, and when subscripted indicates partial differentiation (i.e., $\psi_{n,t} \equiv \partial\psi_n/\partial t$, $\psi_{n,tt} \equiv \partial^2\psi_n/\partial t^2$); the independent (*space*) variable $\vec{r}$ is an S-vector, $\vec{r} \equiv (r_1, \ldots, r_S)$, with S being an *arbitrary* positive integer; $\vec{\nabla}$ respectively $\Delta \equiv \vec{\nabla}\cdot\vec{\nabla}$ denote the standard gradient respectively Laplacian operator in S-dimensional space, $\vec{\nabla} \equiv (\partial/\partial r_1, \ldots, \partial/\partial r_S)$, $\Delta \equiv \sum_{s=1}^{S}\partial^2/\partial r_s^2$; a dot sandwiched among two S-vectors denotes the standard scalar product, $\vec{u}\cdot\vec{v} \equiv \sum_{s=1}^{S}(u_s v_s)$; c is an *arbitrary* constant, which might (but need not) be restricted to be *real* or *imaginary*, $c = \mathrm{i}\gamma$ with γ *real*, thereby transiting from *trigonometric* to *hyperbolic* functions (here and throughout i denotes the imaginary unit); and $\alpha, \beta, \eta, \lambda, \mu$ are *a priori arbitrary* constants—although, as it will be clear from the following, they might also depend on the time or space variables without spoiling the *solvable* character of this system of PDEs.

Particularly interesting—especially in the physical space with $S = 3$—are the *rotation-invariant* nonlinear 'Schrödinger' case characterized by $\alpha = \mathrm{i}$ and $\beta = 0$, the *rotation-invariant* nonlinear 'diffusion' case characterized by $\alpha = 1$ and $\beta = 0$ and the *relativistically invariant* nonlinear 'Klein–Gordon' case characterized by $\alpha = 0, \beta = 1$ and $\eta = -1$. In the first ('Schrödinger') case the dependent variables $\psi_n(\vec{r}, t)$ are necessarily *complex*. In the second and third they might be *real* provided the constants η, λ, μ are *real* and c is *real* or *imaginary*: indeed, in the context of the initial-value problem, the generic (hence nonsingular) solutions are then *real* for all time, $t > 0$, if all the initial data are *real* —namely, the N functions $\psi_n(\vec{r}, 0)$ in the second ('diffusion') case, the $2N$ functions $\psi_n(\vec{r}, 0)$, $\psi_{n,t}(\vec{r}, 0)$ in the third ('Klein–Gordon') case.

In the following section (and in the appendix), the *solvability* of this (system of) PDEs is demonstrated, including the procedure to solve its initial-value problem.

In section 3, we show that the following two (systems of) PDEs,

$$\alpha\left[\tilde{\psi}_{n,t} - \frac{\mathrm{i}\omega}{2}\vec{r}\cdot\vec{\nabla}\tilde{\psi}_n\right] + \eta\Delta\tilde{\psi}_n = 2c\sum_{m=1,m\neq n}^{N}\cot[c(\tilde{\psi}_n-\tilde{\psi}_m)]\eta(\vec{\nabla}\tilde{\psi}_n)\cdot(\vec{\nabla}\tilde{\psi}_m), \qquad (2)$$

respectively

$$\beta\{\tilde{\psi}_{n,tt} - 2\mathrm{i}\omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_{n,t} - \mathrm{i}\omega\tilde{\psi}_{n,t} - \omega^2[(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_n + (\vec{r}\cdot\vec{\nabla})(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_n]\} + \eta\Delta\tilde{\psi}_n$$
$$= 2c\sum_{m=1,m\neq n}^{N}\cot[c(\tilde{\psi}_n-\tilde{\psi}_m)]\{\eta(\vec{\nabla}\tilde{\psi}_n)\cdot(\vec{\nabla}\tilde{\psi}_m)$$
$$+\beta[\tilde{\psi}_{n,t} - \mathrm{i}\omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_n][\tilde{\psi}_{m,t} - \mathrm{i}\omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_m]\}, \qquad (3)$$

where ω is a *positive* constant, $\omega > 0$, are *isochronous*, namely [6] they possess many *isochronous* solutions characterized by the periodicity properties $\tilde{\psi}_n(\vec{r}, t+2T) = \tilde{\psi}_n(\vec{r}, t)$ respectively $\tilde{\psi}_n(\vec{r}, t+T) = \tilde{\psi}_n(\vec{r}, t)$ with $T = 2\pi/\omega$. Note that these systems of PDEs, (2) and (3), are *autonomous* inasmuch as they do *not* feature any explicit dependence on the time coordinate *t*; but they instead do feature an explicit dependence on the space coordinate $\vec{r}$ (they are, however, evidently *covariant*—hence *rotation-invariant*—in S-dimensional space). Note that in this *isochronous* case the dependent variables $\tilde{\psi}_n(\vec{r}, t)$ are necessarily *complex*.

In section 4, we outline the procedure to manufacture instances of solutions of these *solvable* models.

It is plain—see above and below—that, when the constant c is set to zero, $c = 0$, the results of this paper reproduce the main findings reported in [1].

2. Proofs

The starting point of our treatment is the *linear* PDE

$$f\Psi + \alpha\Psi_t + \beta\Psi_{tt} + \eta\Delta\Psi + \lambda\Psi_\psi + \mu\Psi_{\psi\psi} = 0 \tag{4}$$

satisfied by the dependent variable $\Psi \equiv \Psi(\psi; \vec{r}, t)$. Here subscripted variables denote again partial differentiations, for instance $\Psi_\psi \equiv \partial\Psi/\partial\psi$, and the quantities $\alpha, \beta, \eta, \lambda, \mu$ are again *a priori arbitrary* constants, while f is generally (although not necessarily: see the *remark* below) a function of the independent space and time variables, $f \equiv f(\vec{r}, t)$, see below. We moreover focus hereafter on the special solution of this PDE characterized by the *ansatz*

$$\Psi(\psi; \vec{r}, t) = \prod_{n=1}^{N} \frac{\sin\{c[\psi - \psi_n(\vec{r}, t)]\}}{c}, \tag{5a}$$

$$\Psi(\psi; \vec{r}, t) = (2c)^{-N} \sum_{k=0}^{N} \varphi_k(\vec{r}, t) \exp[\mathrm{i}(2k - N)c\psi]. \tag{5b}$$

The compatibility of these two formulae is plain, and it clearly entails

$$\varphi_0(\vec{r}, t) = \mathrm{i}^N \exp[\mathrm{i}c\bar{\psi}(\vec{r}, t)], \qquad \varphi_N(\vec{r}, t) = \mathrm{i}^{-N} \exp[-\mathrm{i}c\bar{\psi}(\vec{r}, t)] \tag{6a}$$

so that

$$\varphi_0(\vec{r}, t)\varphi_N(\vec{r}, t) = 1. \tag{6b}$$

Here and throughout

$$\bar{\psi}(\vec{r}, t) = \sum_{n=1}^{N} \psi_n(\vec{r}, t). \tag{7}$$

It is moreover plain that, given the N quantities ψ_n, the corresponding $N + 1$ quantities φ_k are uniquely defined, being expressed in terms of ψ_n's by formulae (of course consistent with (6)) whose explicit display can be left to the diligent reader; while conversely, given the $N + 1$ quantities φ_k (arbitrarily, except for the restriction (6b)), the N quantities ψ_n are defined up to permutations in the index n, their computation in terms of the quantities φ_k amounting essentially to finding the N zeros of a polynomial of degree N in the variable $z = \exp(2\mathrm{i}c\psi)$. It is moreover obvious from (5a) that the N functions $\psi_n(\vec{r}, t)$ are defined up to the modular ambiguity $\psi_n(\vec{r}, t) + k_n\pi/c$ with the N *integers* k_n *arbitrary* except for the single restriction that their sum be *even*, $\sum_{n=1}^{N} k_n$ = even. And it is evident from (5) that if the n functions $\psi_n(\vec{r}, t)$ are *all real*, the corresponding $N + 1$ functions $\varphi_k(\vec{r}, t)$ are *all real* or *all imaginary* depending whether N is *even* or *odd* if the constant c is *imaginary* (*hyperbolic* case), while they satisfy the $N + 1$ restrictions $\varphi_k(\vec{r}, t) = [\varphi_{N-k}(\vec{r}, t)]^*$ if the constant c is *real* (*trigonometric* case).

As for the compatibility of the *ansatz* (5) with the linear PDE (4), it is guaranteed by the (system of) PDEs satisfied by the quantities $\varphi_k(\vec{r}, t)$, as clearly implied by (4) with (5b):

$$\alpha\varphi_{k,t} + \beta\varphi_{k,tt} + \eta\Delta\varphi_k + [f + \mathrm{i}(2k - N)c\lambda - (2k - N)^2c^2\mu]\varphi_k = 0. \tag{8}$$

Indeed the decoupled character of this system demonstrates its compatibility with the condition that φ_k vanish for $k < 0$ and for $k > N$ (see (5*b*)). The compatibility with (6) is verified below.

It is now fundamental to recognize that the system of (nonlinear and coupled) PDEs satisfied—as implied by (4) with (5*a*)—by the N dependent variables $\psi_n(\vec{r}, t)$ coincides with (1). But this conclusion requires that the quantity $f(\vec{r}, t)$ appearing in (4) and in (8) be identified as follows:

$$f(\vec{r}, t) = c^2\{\beta[\bar{\psi}_t(\vec{r}, t)]^2 + \eta[\vec{\nabla}\bar{\psi}(\vec{r}, t)] \cdot [\vec{\nabla}\bar{\psi}(\vec{r}, t)] + N^2\mu\}, \tag{9}$$

where $\bar{\psi}(\vec{r}, t)$ is defined by (7) hence it satisfies the *linear* PDE

$$\alpha\bar{\psi}_t + \beta\bar{\psi}_{tt} + \eta\Delta\bar{\psi} = N\lambda \tag{10}$$

obtained by summing the system (1) over n from 1 to N and noticing that the double sum over the indices m and n then vanishes due to the antisymmetry of the summand under the exchange of these two indices. Note that this PDE coincides with that obtained by inserting (6*a*) into (8) (with $k = 0$ or $k = N$), thereby confirming the validity of this formula (6*a*).

The derivation via the *ansatz* (5*a*) from (4) with (9) of the system of PDEs (1) satisfied by the independent variables $\psi_n(\vec{r}, t)$ is in principle straightforward, yet we outline it tersely in the appendix.

Let us now indicate how the initial-value problem for the system (1) can be solved. To simplify the presentation we limit our treatment to the case with $\beta = 0$, when the initial data to be assigned are the N functions $\psi_n(\vec{r}, 0)$. The treatment of the case with $\beta \neq 0$ is analogous—except that in this case the initial data to be assigned include moreover the N functions $\psi_{n,t}(\vec{r}, 0)$: its details will be worked out without difficulty by the interested reader.

The solutions to be obtained are the N functions $\psi_n(\vec{r}, t)$. The first step to obtain them is to compute—by integrating the *linear* PDE (10)—the function $\bar{\psi}(\vec{r}, t)$ from the initial datum $\bar{\psi}(\vec{r}, 0)$—itself yielded by the initial data $\psi_n(\vec{r}, 0)$ via (7) (at $t = 0$).

The second step is to compute the function $f(\vec{r}, t)$, as given by formula (9).

The third step is to compute the function $\Psi(\psi; \vec{r}, t)$, by integrating the *linear* PDE (4) from the initial datum $\Psi(\psi; \vec{r}, 0)$ entailed by (5*a*) (at $t = 0$). This task may be facilitated by using the second version, (5*b*), of the *ansatz* (5) and the decoupled system of PDEs (8).

And finally, once $\Psi(\psi; \vec{r}, t)$ has been obtained, the functions $\psi_n(\vec{r}, t)$ can be obtained from (5*a*): note that this formula entails that these N quantities are just the N zeros of $\Psi(\psi; \vec{r}, t)$ considered as a function of the independent variable ψ.

Remark. In the special case with $\lambda = 0$, and if moreover the initial data entail via (7) that $\bar{\psi}(\vec{r}, 0)$ vanishes (and that $\bar{\psi}_t(\vec{r}, 0)$ also vanishes if $\beta \neq 0$), one immediately concludes from (10) that $\bar{\psi}(\vec{r}, t)$ also vanishes, $\bar{\psi}(\vec{r}, t) = 0$, hence (see (9)) that the function f becomes a constant, $f = (Nc)^2\mu$ and (see (6*a*)) $\varphi_0(\vec{r}, t) = \mathrm{i}^N$, $\varphi_N(\vec{r}, t) = \mathrm{i}^{-N}$, while the other quantities $\varphi_k(\vec{r}, t)$ satisfy (see (8)) the following *linear* PDEs with *constant* coefficients:

$$\alpha\varphi_{k,t} + \beta\varphi_{k,tt} + \eta\Delta\varphi_k + 4k(N-k)c^2\mu\varphi_k = 0, \qquad k = 1, \ldots, N-1. \tag{11}$$

3. Isochronous versions

Consider the system (1) with $\beta = \lambda = \mu = 0$ (and with the independent variables $\vec{r}$ and t formally replaced by $\vec{\rho}$ and τ) and set

$$\tilde{\psi}_n(\vec{r}, t) = \psi_n(\vec{\rho}, \tau), \qquad \vec{\rho} \equiv \vec{\rho}(t) = \exp\left(\frac{\mathrm{i}\omega t}{2}\right)\vec{r}, \qquad \tau \equiv \tau(t) = \frac{\exp(\mathrm{i}\omega t) - 1}{\mathrm{i}\omega}. \tag{12}$$

It is then a matter of trivial differentiations to verify that the functions $\tilde{\psi}_n(\vec{r}, t)$ satisfy the system of PDEs (2).

A completely analogous treatment yields the system (3), except that one must now start with the system (1) with $\alpha = \lambda = \mu = 0$ and set

$$\tilde{\psi}_n(\vec{r}, t) = \psi_n(\vec{\rho}, \tau), \qquad \vec{\rho} \equiv \vec{\rho}(t) = \exp(\mathrm{i}\omega t)\vec{r}, \qquad \tau \equiv \tau(t) = \frac{\exp(\mathrm{i}\omega t) - 1}{i\omega}. \tag{13}$$

And the relations among the functions $\tilde{\psi}_n(\vec{r}, t)$ and $\psi_n(\vec{\rho}, \tau)$, (12) respectively (13), together with the *solvable* character of the system (1), justify [6] the assertions made in the introductory section about the *isochronous* characters of the systems of PDEs (2) respectively (3).

4. Examples

In this section, we outline the procedure to manufacture solutions $\psi_n(\vec{r}, t)$ of the systems of evolution PDEs (1), (2) and (3). The most interesting exhibition of such solutions is via *animations*, which we plan to display in due time in an appropriate electronic journal.

For simplicity, we limit our consideration here to the special case with $\lambda = 0$ and $\bar{\psi}(\vec{r}, t) = 0$, when $\varphi_0(\vec{r}, t) = \mathrm{i}^N, \varphi_N(\vec{r}, t) = \mathrm{i}^{-N}$, and the other $N - 1$ quantities $\varphi_k(\vec{r}, t)$ satisfy the *linear* PDEs with *constant* coefficients (11) (the alert reader will easily figure out how to proceed in the more general case without these restrictions). Manufacturing *explicit* solutions $\varphi_k(\vec{r}, t)$ of these PDEs, (11), is then a trivial task, and by inserting them into (5*b*) one obtains an *explicit* expression of the function $\Psi(\psi; \vec{r}, t)$. The computation of the corresponding solutions $\psi_n(\vec{r}, t)$ amounts then to evaluating the zeros of $\Psi(\psi; \vec{r}, t)$ considered as a function of the variable ψ: hence essentially—see (5*a*), and the remarks written above following (7)—to finding the N zeros of a polynomial of degree N in the variable $z = \exp(2\mathrm{i}c\psi)$, the coefficients of which depend in an *explicitly known* manner on the space and time coordinates $\vec{r}$ and t.

And to every solution $\psi_n(\vec{r}, t)$ of the (system of) evolution PDEs (1) with $\beta = \lambda = \mu = 0$ respectively with $\alpha = \lambda = \mu = 0$ there corresponds via (12) respectively via (13) a solution $\tilde{\psi}_n(\vec{r}, t)$ of the *isochronous* (systems of) evolution PDEs (2) respectively (3).

Appendix. Derivation of the system of PDEs (1)

Logarithmic differentiation of (5*a*) yields

$$\Psi_\psi(\psi; \vec{r}, t) = \Psi(\psi; \vec{r}, t) c \sum_{n=1}^{N} \cot\{c[\psi - \psi_n(\vec{r}, t)]\}, \tag{A.1}$$

and an additional differentiation yields

$$\Psi_{\psi\psi} = \Psi c^2 \left\{ -\sum_{n=1}^{N} \{1 + \cot^2[c(\psi - \psi_n)]\} + \sum_{m,n=1}^{N} \cot[c(\psi - \psi_n)] \cot[c(\psi - \psi_m)] \right\}, \tag{A.2a}$$

$$\Psi_{\psi\psi} = \Psi c^2 \left\{ -N + \sum_{m,n=1;m\neq n}^{N} \cot[c(\psi - \psi_n)] \cot[c(\psi - \psi_m)] \right\}, \tag{A.2b}$$

$$\Psi_{\psi\psi} = \Psi c^2 \left\{ -N^2 + 2\sum_{n=1}^{N} \cot[c(\psi - \psi_n)] \sum_{m=1,m\neq n}^{N} \cot[c(\psi_n - \psi_m)] \right\}. \tag{A.2c}$$

The derivation of the second version of this formula from the first is obvious, and the derivation of the third version from the second is also plain by using the trigonometric identity

$$\cot x \cot y = -1 - (\cot x - \cot y)\cot(x - y). \tag{A.3}$$

Likewise one obtains the formulae

$$\Psi_t(\psi;\vec{r},t) = \Psi(\psi;\vec{r},t)c\sum_{n=1}^{N}\cot\{c[\psi - \psi_n(\vec{r},t)]\}[-\psi_{n,t}(\vec{r},t)], \tag{A.4}$$

$$\Psi_{tt} = \Psi c\left\{-c\bar{\psi}_t^2 + \sum_{n=1}^{N}\cot[c(\psi - \psi_n)] \times\left\{-\psi_{n,tt} + 2c\sum_{m=1,m\neq n}^{N}\cot[c(\psi_n - \psi_m)]\psi_{n,t}\psi_{m,t}\right\}\right\}, \tag{A.5}$$

$$\vec{\nabla}\Psi(\psi;\vec{r},t) = \Psi(\psi;\vec{r},t)c\sum_{n=1}^{N}\cot\{c[\psi - \psi_n(\vec{r},t)]\}[-\vec{\nabla}\psi_{n,}(\vec{r},t)], \tag{A.6}$$

$$\Delta\Psi = \Psi c\left\{-c(\vec{\nabla}\bar{\psi})\cdot(\vec{\nabla}\bar{\psi}) + \sum_{n=1}^{N}\cot[c(\psi - \psi_n)] \times\left\{-\Delta\psi_n + 2c\sum_{m=1,m\neq n}^{N}\cot[c(\psi_n - \psi_m)](\vec{\nabla}\psi_n)\cdot(\vec{\nabla}\psi_m)\right\}\right\}. \tag{A.7}$$

The insertion into (4) of these formulae, (A.1), (A.2*c*), (A.4), (A.5), (A.6), (A.7), yields (1) and (9).

References

[1] Calogero F and Sommacal M 2006 Solvable nonlinear evolution PDEs in multidimensional space *SIGMA* **2** 088 (17 pp) (*Preprint* nlin.SI/0612019)
[2] Calogero F 1994 A class of C-integrable PDEs in multidimensions *Inverse Problems* **10** 1231–4
[3] Calogero F 2001 *Classical Many-Body Problems Amenable to Exact Treatments (Lecture Notes in Physics Monograph* m 66*)* (Berlin: Springer)
[4] Calogero F and Françoise J-P 2000 A novel solvable many-body problem with elliptic interactions *Int. Math. Res. Not.* **15** 775–86
[5] Calogero F, Françoise J-P and Sommacal M Solvable nonlinear evolution PDEs in multidimensional space involving elliptic functions (in preparation)
[6] Calogero F Isochronous systems *Monograph* 200 pp (in preparation)